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Computation of Flight Mechanics Parameters Using Continuation Techniques

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A new approach is presented for the computation of continuation diagrams. It differs in principle from the usual approach of continuing the equilibrium points by considering one or more of the controls as continuation parameters and with the other controls held fixed. The novel approach addresses the problem of flight dynamics from the pilot point of view by allowing all the controls to be freely determined. At the same time some of the states are so constrained that along the continuation curve a particular pilot trim condition is maintained.

Nomenclature

$\boldsymbol{f}$	=	right-hand side of aircraft rigid-body equations
$\boldsymbol{f}_{rc}$	=	eighth-order subset of equations of motion restricted to level flight
$\boldsymbol{f}_1, \boldsymbol{f}_r$	=	eighth-order subset of equations of motion
p, q, r	=	aircraft body axis roll, pitch, and yaw rates
$\boldsymbol{u}$	=	control vector, dimension p
V_T	=	total aircraft velocity
X, Y, H	=	aircraft position with respect to Earth-fixed inertial axes
$\boldsymbol{x}$	=	state vector, dimension n
α	=	angle of attack
β	=	angle of sideslip
γ	=	flight-path angle
$\delta_e, \delta_a, \delta_r$	=	elevator, aileron, and rudder control surface deflections
δ_p	=	throttle position
$\boldsymbol{\lambda}$	=	parameter vector
ϕ, θ, ψ	=	Euler angles orienting aircraft body axes with respect to inertial axes

Subscript

0	=	equilibrium value of the variable

I. Introduction

LINEAR control synthesis techniques rely on the linearized equations of motion of the aircraft about equilibrium points,[1] that is, points at which state derivatives are equal to zero. It is common practice to generate hundreds of such linear models during practical design of control systems for aircraft using software tools.[2] Because these linear models are expected to represent the nonlinear aircraft dynamics in various stages of flight, it is desirable to generate these models for different trim conditions. These trim conditions correspond closely to the operational conditions (1-g level flight, level turn, high-g climb, turns, etc.) encountered in aircraft flight. Certain parameters are kept fixed, for example, flight-path angle, turn rate, Mach number, altitude, etc., when using standard optimization methods to obtain the equilibrium point. In this paper it is shown how this computation can be performed efficiently by the use of continuation methods.

Numerical continuation methods have been used successfully for a long time in the study of aircraft nonlinear dynamics.[3–5] In particular, continuation methods have been used to track the equilibrium states of aircraft into the nonlinear flight regimes. The bifurcation analysis of aircraft equilibrium states is an area of intense interest at the present time.[6–8] A comprehensive introduction to these methods and their application to aircraft dynamics is presented in Ref. 9. The importance of bifurcation analysis lies in its ability to give a global picture of the stability and control characteristics of the aircraft under investigation. The use of bifurcation analysis makes it possible to append the control law designed for the aircraft (which may be linear or nonlinear) with the open-loop aircraft equations of motion and to study the properties such as the stability of the combined system (aircraft and control law). This aspect of the analysis has resulted in the use of bifurcation analysis for the development of control design strategies for aircraft in the nonlinear high-angle-of-attack regimes of flight.[10–13] The literature up to the present time has restricted itself to computing the equilibrium continuation by allowing variation in only one or two control parameters, for example, the elevator, the aileron, or the rudder. This is a natural outcome of the desire to analyze the stability of the aircraft using the Lyapunov method (Ref. 14). If the controls are held fixed the stability of the aircraft can be studied as a problem in initial condition disturbances. When the controls are considered as parameters, it is possible to study the relation between stability and control by means of continuation.

In contrast to the method of analysis, the reality is somewhat different. When flying an aircraft a pilot uses all of the controls possible to maintain a certain equilibrium condition, for example, level flight. Thus, the pilot or automatic control system is concerned only with the behavior of certain aircraft variables, seeking to maintain them within certain limits or to cancel them by adequate control movements. For example, in case the pilot intends to maintain straight level flight, the pilot will use all of the controls to keep the airspeed constant, flight path on the horizon, and wings level. A theory accounting for such a situation is discussed in Chapter 4 of Ref. 14. It is called the problem of partially controlled motions. This situation is not captured by the current approach to the computation of the bifurcation curves. It is more natural and appropriate for aircraft dynamics studies to consider the variation of all controls necessary to achieve a given type of trim. This can be achieved by combining the ideas of Ref. 1 with the numerical continuation approach.

The remainder of this paper is organized as follows. Section II describes the assumptions and the aircraft model used for the computations. Section III describes the usual approach adopted in bifurcation analysis. A typical result of such an approach is presented in the plots of the continuation curves. Section IV describes the new approach

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along with the corresponding plots of the continuation curves. The straight and level trim is used to motivate this. The general principle of this new approach is also discussed. In Sec. V, other important issues such as choice of starting point for continuation and the effect of decoupled equations on the continuation curves are discussed.

II. Analysis Assumptions and Aircraft Model

A common feature of the continuation techniques is the study of the equilibrium points of the nonlinear aircraft equations represented by the smooth mapping $\boldsymbol{f}$: $R^{n+p} \rightarrow R^n$,

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) = 0 \tag{1}$$

where $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ represents the nonlinear functions in the aircraft states $\boldsymbol{x}$ (dimension n) and control inputs $\boldsymbol{u}$ (dimension p). The state and control vectors are given by

$$\boldsymbol{x} = (V_T, \alpha, \beta, p, q, r, \phi, \theta, \psi, X, Y, H), \qquad \boldsymbol{u} = (\delta_e, \delta_a, \delta_r, \delta_p) \tag{2}$$

The equations of motion represented by Eq. (1) are essentially the rigid-body equations of motion on a flat Earth specialized to fixed wing aircraft. It is assumed that the mapping $\boldsymbol{f}$ is smooth and has as many derivatives as the subsequent discussion requires. Useful equilibriums are obtained by considering the first eight state equations:

$$\boldsymbol{x} = (V_T, \alpha, \beta, p, q, r, \phi, \theta), \qquad \boldsymbol{u} = (\delta_e, \delta_a, \delta_r, \delta), \qquad \boldsymbol{\lambda} = (H) \tag{3}$$

Note that the altitude H appears as a parameter on the right-hand sides of this reduced system of equations. It is held fixed ($H = H_0$) during the computations unless designated as a continuation parameter. To clarify the distinction between parameters, states, and controls, a formal definition of parameters is given in Sec. IV. It is usual to study the aircraft high-angle-of-attack dynamics with the help of the eighth-order model. The mapping $\boldsymbol{f}_1$: $R^m \rightarrow R^n$ represents this model:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}_1(\boldsymbol{x}, \boldsymbol{u}, \lambda) = 0 \tag{4}$$

where $m = n + p + 1$, $n = 8$, and $p = 4$. In the following sections we shall consider the eighth-order model for analysis unless otherwise stated.

The aircraft model used in the computations is the F-16 aerodynamic model based on global multivariate polynomials.[15] This aerodynamic database was generated from wind-tunnel tests and is presented in Ref. 16. The representation by polynomials results in a compact and smooth functional representation for the database. This is ideal for the continuation methods, although they work fairly well for the tabular database. The aerodynamic model is low-speed and high angle of attack. Therefore, compressibility effects are not accounted for in the aerodynamic database. A suitable engine model was also provided along with the aerodynamic data description of the aircraft. The engine flames out at high angles of attack unless special provisions are made. In the high-angle-of-attack analysis, it is quite usual to neglect the engine dynamics altogether and set the thrust value to zero in the equations of motion. In this work, the engine model is used where appropriate.

The continuation software used for calculation of the equilibrium curves in the following sections is based on an Euler–Newton type predictor–corrector algorithm.[17] The predictor is based on an ordinary differential equation (ODE) solver, whereas the corrector converges iteratively in a direction perpendicular to the computed tangent of the continuation curve. A relative tolerance of 10^{-3} was used for step size control of the ODE solver, whereas a value of 10^{-4} was used as the absolute tolerance for convergence of the corrector. The tangent computation is based on the QR decomposition algorithm with Givens rotations (Ref. 17). A condition number estimate of the Jacobian of the mapping was also computed and monitored in case the algorithm encountered difficulty. The algorithm did encounter difficulties at high angles of attack and when the equations become singular, for example, when pitch attitude approaches 90 deg. A pseudoarc length parameterization was used for the continuation. Bifurcations were detected by a change of orientation. The algorithm jumped over these simple bifurcations. No attempt was made to compute the transverse branches.

III. Continuation Technique

A starting point is needed for computation of the continuation curve. To obtain such a starting point, the usual method is to fix all controls (elevator, aileron, rudder, and throttle) and parameters (altitude) in Eq. (4) to suitable values. The resulting map is $\boldsymbol{f}_1|_{u=u_{e0},\lambda=\lambda_0}$: $R^n \rightarrow R^n$. Standard optimization methods can be used for obtaining a fixed point of such a map. Let the equilibrium point so obtained be labeled as $(\boldsymbol{x}_{e0}, \boldsymbol{u}_{e0}, \boldsymbol{\lambda}_0)$, then

$$\boldsymbol{f}_1(\boldsymbol{x}_{e0}, \boldsymbol{u}_{e0}, \boldsymbol{\lambda}_0) = 0 \tag{5}$$

where the control and the parameter vectors have the equilibrium values

$$\boldsymbol{u}_{e0} = (\delta_{e0}, \delta_{a0}, \delta_{r0}, \delta_{p0}), \qquad \boldsymbol{\lambda}_0 = (H_0) \tag{6}$$

In most of the existing literature, continuation is carried out with respect to either one parameter or two parameters at a time. In the one-parameter case, any three of the control variables are fixed, for example, the aileron, rudder, and throttle are fixed at the values δ_{a0}, δ_{r0}, and δ_{p0}, respectively, and the remaining control, the elevator δ_e, is allowed to vary. The altitude H, which appears as a parameter in the equations, is held fixed ($H = H_0$). This gives a map $\boldsymbol{f}_r$: $R^{n+1} \rightarrow R^n$:

$$\boldsymbol{f}_r(V_T, \alpha, \beta, p, q, r, \phi, \theta, \delta_e) = \boldsymbol{f}_1(x, u, \lambda)|_{\delta_a=\delta_{a0},\delta_r=\delta_{r0},\delta_p=\delta_{p0},H=H_0} = 0 \tag{7}$$

This function can be continued provided certain smoothness and rank conditions are satisfied.[17] The curve obtained by computing a solution branch of Eq. (7) beginning with an initial trim flight condition is shown in Fig. 1. This computation was obtained for a zero throttle and −2.5-deg elevator setting. The aileron and rudder were set to zero.

A lot of useful information can be obtained by the study of Fig. 1. In particular, it is possible to compute the stability regions of initial condition disturbances for these equilibrium points and, thus, determine the region in joint state and control space where the system is stable. There is, however, a problem in interpreting Fig. 1. The initial point $(\boldsymbol{x}_{e0}, \boldsymbol{u}_{e0}, \boldsymbol{\lambda}_0)$, marked with an asterisk in Fig. 1, appears to be a straight level flight trim. This point actually has nonzero pitch angle. However, along the curve this trim condition is not maintained because the flight-path angle is not constant (see pitch attitude and angle of attack).

It is possible to start the continuation from a point that is a trim state in the sense of Ref. 1, but there is no reason to expect that the other points obtained on the curve will also represent a particular type of equilibrium. Note that the continuation curve generated in Fig. 1 is perfectly valid, but difficult to interpret in terms of our conventional understanding of trimmed flight. A more appropriate continuation would be that of the straight level flight trim condition. This is also a natural setting for obtaining the linear models required for the design of the control laws in the operational flight regime. It is typical of a pilot to maintain a given type of trim (level flight, steady turn, steady climb, etc.) by manipulating all available controls rather than fix one or more of the controls at a given setting. A different approach is needed for performing such computations. The proposed new approach is discussed in the next section.

IV. New Approach

It is instructive to compare Eq. (4) with the function given by Eq. (7). The altitude H is considered as a parameter in both cases. The function $\boldsymbol{f}$ has $8 + 4 + 1$ independent variables, whereas $\boldsymbol{f}_r$ has $8 + 1$. The latter is obtained from the former by simply fixing the remaining independent variables. If some other suitable independent variables (out of the $8 + 4 + 1$) are held fixed, it may be possible to generate another function, which can be continued in a manner

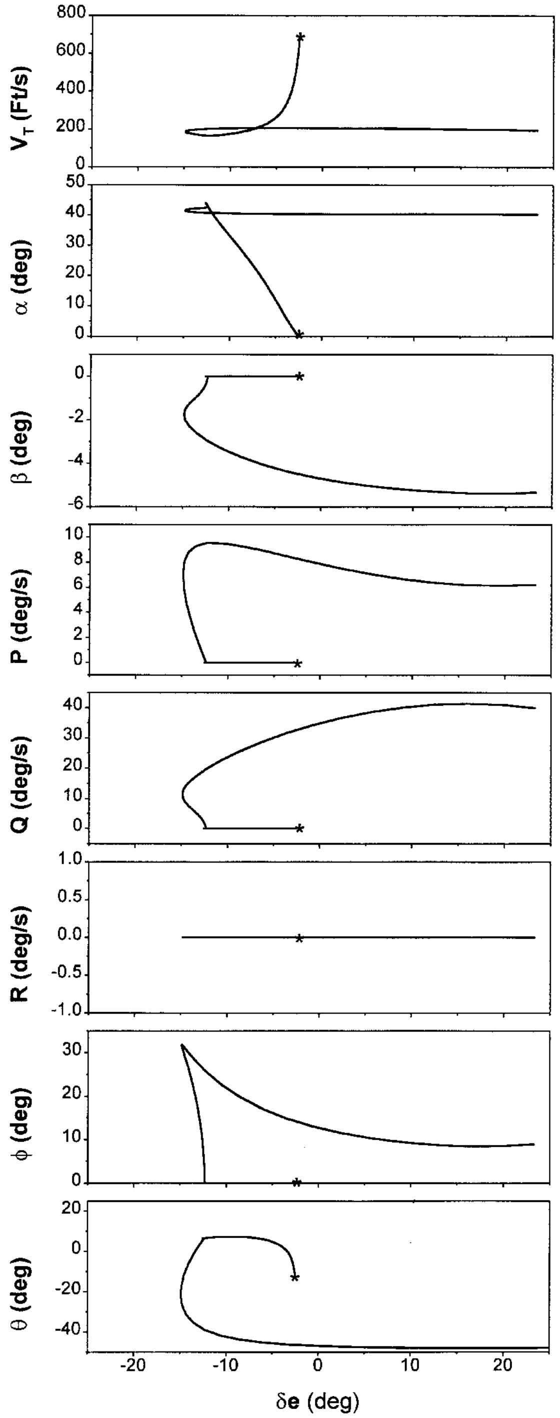

Fig. 1 Continuation diagram for the mapping f_r.

similar to f_r. For example, straight and level flight is a trim point of Eq. (4) with the additional constraints

$$V_T = V_0(= \text{const}), \qquad r = 0, \qquad \phi = 0$$

$$\theta = \alpha + \tan^{-1}\left(\frac{\sin\gamma/\cos\beta}{\sqrt{1-(\sin\gamma/\cos\beta)^2}}\right), \qquad H = H_0(= \text{const}) \tag{8}$$

where γ is the flight-path angle with the horizon. It is clear from Eq. (8) that for level flight the four states, V_T, r, ϕ, and θ, are determined in terms of the other states, α and β, and the parameters H_0, V_0, and γ. If we further require that the flight be at a constant altitude (straight level flight), then the parameter γ is a constant (=0), and the last equation reduces to $\theta = \alpha$.

Table 1 Details of the mapping f_r[a]

State equations	Free variables for trim,[b] states x	Constraints
$\dot{V}_T$	V_T	$\delta_a = \delta_{a0}$
$\dot{\alpha}$	α	$\delta_r = \delta_{r0}$
$\dot{\beta}$	β	$\delta_p = \delta_{p0}$
$\dot{p}$	p	$H = H_0$
$\dot{q}$	q	——
$\dot{r}$	r	——
$\dot{\phi}$	ϕ	——
$\dot{\theta}$	θ	——

[a]Parameters λ for continuation are δ_e.
[b]There are no controls $\boldsymbol{u}$.

Table 2 Details of the mapping f_{rc}[a]

State equations	Free variables for trim: States x	Controls u	Constraints
$\dot{V}_T$	——	δ_e	——
$\dot{\alpha}$	α	δ_a	——
$\dot{\beta}$	β	δ_r	——
$\dot{p}$	p	δ_p	$H = H_0$
$\dot{q}$	q	——	——
$\dot{r}$	——	——	$r = 0$
$\dot{\phi}$	——	——	$\phi = 0$
$\dot{\theta}$	——	——	$\theta = \alpha$

[a]Parameters λ for continuation are V_T.

A different function f_{rc} can be defined in a manner similar to f_r by allowing the set (V_T, α, β, p, q, δ_e, δ_a, δ_r, δ_p) of 8 + 1 independent variables to be free. The remaining three states, r, ϕ, and θ are forced to satisfy the constraints in Eq. (8) with γ at a fixed value (typically $\gamma = 0$). Note that the parameter V_T is treated as the continuation parameter here. Then we have the map f_{rc}: $R^{n+1} \to R^n$:

$$f_{rc}(V_T, \alpha, \beta, p, q, \delta_e, \delta_a, \delta_r, \delta_p) = f_1(x, u, \lambda)|_{r=0,\phi=0,\theta=\alpha,H=H_0} = 0 \tag{9}$$

The continuation curve of this new function f_{rc} has the property that, along it, the straight level flight condition is also satisfied. This is the desired continuation curve for straight and level flight. It remains to choose a method of obtaining a suitable starting point for the computation of this curve. The straight level trim defined in Ref. 1, from where the constraint equations (8) were obtained, is the appropriate method to use. This is achieved by fixing the total velocity in Eq. (9) to obtain a map $f_1|_{r=0,\phi=0,\theta=\alpha,H=H_0,V_T=V_0}$: $R^n \to R^n$.

To make the difference between the two functions f_r and f_{rc} clear, Tables 1 and 2 are provided, respectively. Tables 1 and 2 list the equations, the state and control variables, and the parameters for the respective functions. The continuation curve of the mapping f_{rc} is defined under same smoothness and rank conditions required for continuation of the mapping f_r (Ref. 17).

Figure 2 shows the continuation curve obtained by this method. A comparison with Fig. 1 indicates many important differences. The angular rates, sideslip angle, and bank angle are practically zero throughout the angles-of-attack range. The lowest thrust setting required to hold straight level flight at this altitude (sea level) can be read off directly (point A in Fig. 2) as 5-deg power lever angle (which is close to flight idle for this aircraft). This corresponds to about a 4-deg angle of attack. An interesting feature of the curve generated from f_{rc} is that the static stability of the aircraft can be seen immediately from the plot of angle of attack against the elevator. Because this aircraft is stable (static margin = 5% chord), the slope of this plot is negative. Other useful information that can be obtained is the controllability of the aircraft at high angles of attack.[18]

It is possible to apply this idea for trim types other than straight level flight.[1] The general idea proposed is as follows:

1) Choose a suitable subset of the rigid-body equations, for example, $f_1(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\lambda})$. Here $\boldsymbol{\lambda}$ is a set of parameters, which appear on the right-hand sides of f_1, but are not the states or control variables of this chosen subset. They are to be held fixed unless used

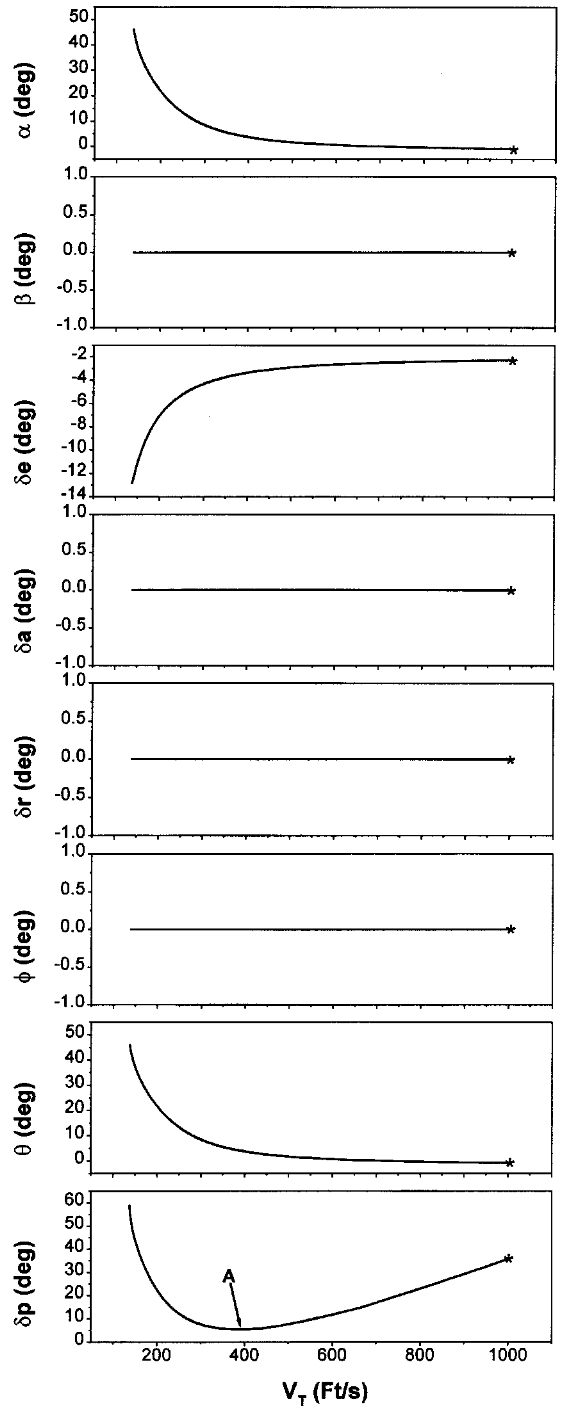

Fig. 2 Continuation diagram for the mapping f_{rc}.

for continuation later, for example, the eighth-order set of equations discussed in Sec. II.

2) Choose a particular type of trim from Ref. 1. This means choosing the appropriate trim constraint equations $c(x, \lambda) = 0$. There must be exactly as many constraint equations as the number of controls, that is, $\dim[c(x, \lambda)] = p$. Here λ is now the set of parameters that is fixed for that trim as per Ref. 1 plus those held fixed in step 1. For example, for the level flight case chosen in this section, Eqs. (8) represent the constraints, whereas the altitude, total velocity, and the flight-path angle γ are the parameters. If $\gamma = 0$, we have the straight and level flight case.

3) Choose a particular parameter from the set λ, for example, λ_1, as the continuation parameter. Then we have $\lambda = \{\lambda_1, \lambda_2\}$, where λ_2 represent the remaining parameters. Fix all of the remaining parameters, for example, $\lambda_2 = \lambda_{20}$. The map $f_1(x, u, \lambda)$ constrained by $c(x, \lambda_1, \lambda_{20}) = 0$ defines another mapping $f_1|_{c(x,\lambda_1,\lambda_{20})=0} = f_{rc}$ such that f_{rc}: $R^{n+1} \to R^n$. For example, in the straight and level flight case, the altitude and flight-path angle γ correspond to the fixed parameters, and the total velocity is the continuation parameter.

4) Obtain a starting point $(\boldsymbol{x}_{e0}, \boldsymbol{u}_{e0})$ of f_{rc} by first fixing the continuation parameter λ_1 at a value for example, λ_{10}, that is, solve for $(\boldsymbol{x}, \boldsymbol{u})$ given $f_{rc}(\boldsymbol{x}, \boldsymbol{u}, \lambda_{10}, \lambda_{20}) = 0$. This is a mapping, which is $R^n \to R^n$; hence, standard optimization methods can be used for this purpose. For example, any point where straight level flight condition is satisfied can be used as a starting point.

5) Continue the solution $(\boldsymbol{x}_{e0}, \boldsymbol{u}_{e0}, \lambda_1)$ in the state, control, and parameter space. Note that the constraining equation c in this discussion is applied only on the states and not on the controls. To make the discussion clear, a definition of parameters is now given.

Definition: The set of parameters $\boldsymbol{\lambda}$ consists of all of those quantities appearing on the right-hand side of the system of equations considered for analysis (represented by the set of functions f_1), which have to be fixed to a nonzero real number to make the mapping f_1: $R^m \to R^n$, $(m \geq n)$ into a restricted mapping $f_1|_{\lambda=\text{const}}$: $R^n \to R^n$.

V. Discussion and Comparison Between the Two Methods

Let us examine the function f_1: $R^m \to R^n$, given by Eq. (4). In this case, $m = 8 + 4 + 1$ and $n = 8$. The set of nonlinear equations given by

$$f_1(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\lambda}) = 0 \tag{10}$$

is underdetermined. In Sec. III and in the preceding discussion, only $8 + 1$ out of the $8 + 4 + 1$ independent variables were allowed to vary. The result of continuation was a one-dimensional manifold, that is, a curve. If all of the $8 + 4 + 1$ independent variables are allowed to vary, we shall have a manifold defined by Eq. (10) of dimension five. Then the continuation curves generated by f_r and f_{rc} are particular but different sections of this five-dimensional manifold. An example of such a computation is seen in Ref. 19 for the Alpha-Jet. From the interpretation point of view, it seems easier to understand the curve generated by f_{rc} as opposed to f_r.

The linear models required for the control law design can be simply obtained by generating the appropriate first derivatives along the curve f_{rc}. Thus, the linear model is obtained as

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$$

where

$$A = \frac{\partial f_1}{\partial \boldsymbol{x}}, \qquad B = \frac{\partial f_1}{\partial \boldsymbol{u}}$$

are the system and input matrices, respectively. Note that although the nominal trim point is along f_{rc}, the derivatives are taken on the unconstrained function f_1.

In many conditions, the aircraft equations can be decoupled into longitudinal and lateral modes. Then the longitudinal state and control variables $(V_T, \alpha, q, \theta, \delta_e, \delta_p)$ affect the longitudinal equations f_{lon}: $R^6 \to R^4$ alone, whereas the lateral/directional state and control variables $(\beta, p, r, \phi, \delta_a, \delta_r)$ affect the lateral/directional equations f_{lat}: $R^6 \to R^4$. The altitude is a parameter that is fixed for the analysis $(H = H_0)$. It is possible to write Eq. (10) in decoupled form as

$$f_1(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\lambda})|_{H=H_0} = \begin{bmatrix} f_{\text{lon}}(V_T, \alpha, q, \theta, \delta_e, \delta_p) \\ f_{\text{lat}}(\beta, p, r, \phi, \delta_a, \delta_r) \end{bmatrix} = 0 \tag{11}$$

In the low-angle-of-attack regions, the equations of motion are well represented by Eq. (11). Consider the effect of using continuation on the corresponding functions f_r [Eq. (7)] and f_{rc} [Eq. (9)]. In the first case, the aileron, rudder, and throttle (δ_a, δ_r, and δ_p) are held fixed, and the elevator δ_e is treated as the continuation parameter. In the second case, roll rate, bank angle, and pitch attitude, r, ϕ, and θ, are fixed, and all of the controls are free. Clearly, the two methods will result in different continuation curves even for this case. Figure 3 shows a comparison of between the results obtained by the two different methods outlined in Secs. III and IV. This clearly shows

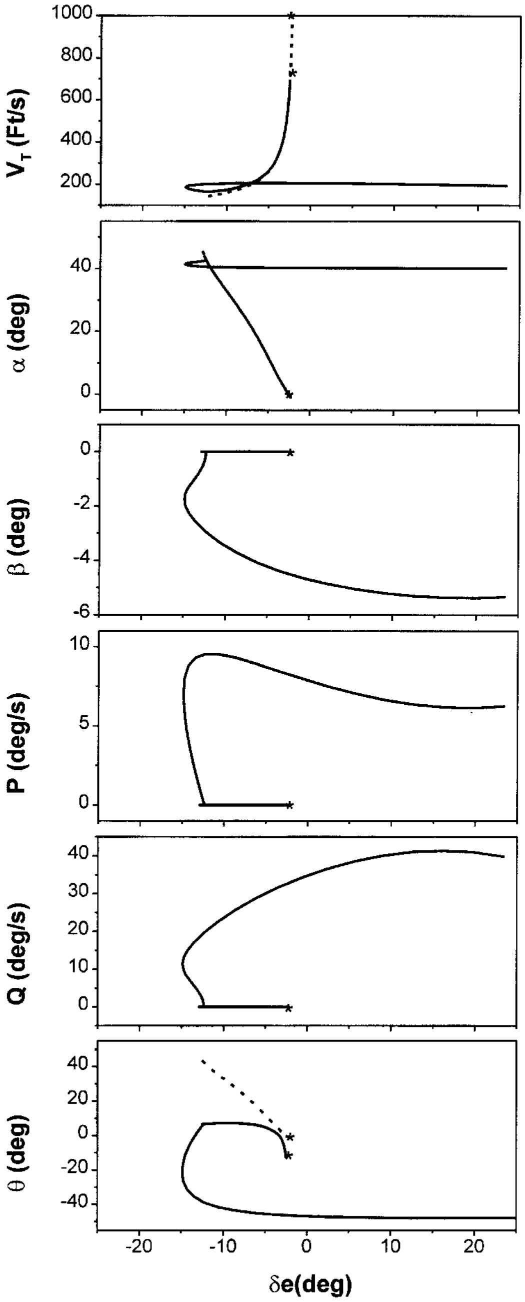

Fig. 3 Comparison between the result of Figs. 1 and 2 (——, f_r, and - - - -, f_{rc}).

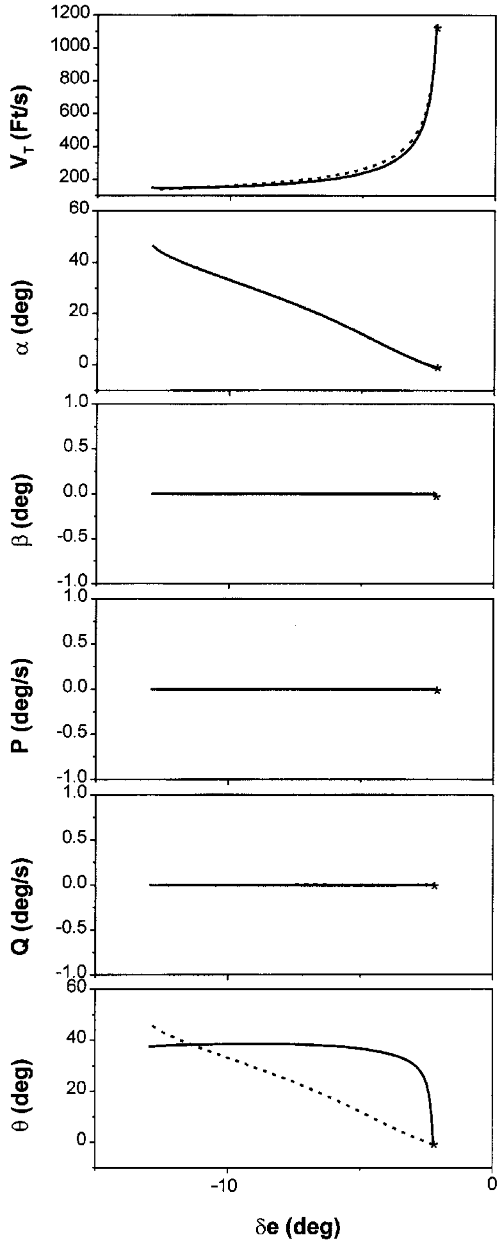

Fig. 4 Comparison between continuation diagrams of f_r (——) and f_{rc} (- - - -) with same initial condition *.

that differences exist even at low angles of attack. In the longitudinal variables, this difference arises due to the variation of speed and throttle. However, the differences can be quite small in the primary longitudinal variables α, β, p, q, and r due to weak dependence of the equations on the total velocity. Also, it is expected that the lateral equations will be generally satisfied in an approximate manner if all of the lateral states and controls are close to zero, due to symmetry considerations. This has led many analysts[20] to remove the $\dot{V}_T$ (total velocity rate) equation altogether and consider the reduced system for analysis.

Another aspect of these two curves is worth some consideration. The problem of continuation of a mapping $\boldsymbol{g}$: $R^{n+1} \rightarrow R^n$ defines an initial value problem[17] given by

$$\boldsymbol{g}'(\omega)\dot{\omega} = 0, \qquad \omega(0) = (\boldsymbol{x}_{e1}, \boldsymbol{u}_{e1}, \boldsymbol{\lambda}_1) \tag{12}$$

where $\boldsymbol{x}_{e1}$, $\boldsymbol{u}_{e1}$, and $\boldsymbol{\lambda}_1$ correspond to the initial values of those state variables, controls, and the parameter (out of the total set of states, controls, and parameters) that are allowed to vary along the continuation curve generated by $\boldsymbol{g}$. The term $\boldsymbol{g}'(\omega)$ in Eq. (12) represents the Jacobian of the mapping with respect to all of these free variables, and $\dot{\omega} = \mathrm{d}\omega/\mathrm{d}s$ (with $\|\dot{\omega}\| = 1$) is the unit tangent to the continuation curve. In light of this, note the results of continuation of the two curves $\boldsymbol{f}_r$ and $\boldsymbol{f}_{rc}$ from the same initial point. In Fig. 3, the two curves were generated using different initial starting values for the two different continuation curves. The strategy used to obtain a starting value for continuation of the map $\boldsymbol{f}_r$ is distinctly different from that used for the map $\boldsymbol{f}_{rc}$. Note the effect of using a straight and level starting value (as used for $\boldsymbol{f}_{rc}$) in the continuation computations of $\boldsymbol{f}_r$. It is obvious that any point on $\boldsymbol{f}_{rc}$ can be used as a starting point for $\boldsymbol{f}_r$, although the converse is not true. This does not imply that all points on $\boldsymbol{f}_{rc}$ also lie on $\boldsymbol{f}_r$. The results of starting from the same initial condition are shown in Fig. 4. Again it is seen that the primary longitudinal variables are quite similar for both methods. However, differences exist in the pitch attitude.

VI. Conclusions

A novel approach has been presented for the computation of continuation diagrams, which differs considerably from the conventional control fixed approach. The advantages of the new approach are as follows:

1) The trims can be defined in the conventional way in terms of constraints on the state variables, for example, straight level flight, level turning flight, and so on.

2) The continuation curves directly give open-loop controllability information for a particular trim. In other words it is possible to obtain directly the region in state and control space in which a particular trim can be sustained (open loop) with the existing control surface authority. This can be used in the design phase of a planform.

3) The approach is directly useful for the generation of linear models for control design. It considerably simplifies the generation and study of such models.

4) The longitudinal static stability characteristics of the planform can be seen immediately from the plot of the trim angle of attack against the elevator deflection.

There are certain drawbacks inherent in the proposed method. Because only the trim points are being computed, it is not possible to obtain the curves in regions where a conventional trimmed flight is not possible. For example, as seen in the paper, it is not meaningful to exceed any of the limits imposed by the thrust and aerodynamic control surfaces when continuing the straight and level flight trim. In contrast, in the conventional approach where one control surface is varied at a time, it is possible to continue extreme flight conditions such as spin into the low-angle-of-attack region without violating the constraints on the controls. Also by rigidly constraining the equilibrium curve to one type of equilibrium, for example, level flight, information about other nearby equilibriums is lost. It is suggested that the two approaches be used in a complementary fashion. The conventional approach can be used at extreme flight conditions, while the novel approach can be used within the bounds of trimmed flight. A judicious combination of the two approaches can yield improved understanding of the aircraft planform under analysis.

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